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On Certain Best Constants for Bernstein-Type Operators[1]

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We discuss the generalized version of a best-constant problem raised by Z. Li in a note which recently appeared in *Journal of Approximation Theory*. Some best

operators; preservation properties.

1. INTRODUCTION

In a recent paper by Li [11], it is shown (among other things) that

$$C := \sup_{n \geqslant 1} \sup_{0 < x \leqslant 1} \sup_{\omega \in \Omega} \frac{B_n \omega(x)}{\omega(x)} = 2, \tag{1}$$

where $B_n \omega$ is the nth Bernstein polynomial of the function ω,

$$B_n \omega(x) := \sum_{k=0}^{n} \omega(k/n)\, p_{n,k}(x), \qquad p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},$$

and Ω is the set of all continuous moduli of continuity on $[0, 1]$, i.e., the set of all real nonidentically zero continuous functions on $[0, 1]$ which vanish at 0 and are nondecreasing and subadditive.

It should be said that, since $B_n(\Omega) \subset \Omega$ $(n \geqslant 1)$ (cf. [11]), and each $\omega \in \Omega$ coincides with its usual modulus of continuity, such a result can be viewed as a consequence of [4, Theorem 9 and the subsequent Remark (ii)].

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On the other hand, in the same paper [11], the problem of determining whether or not $C^* < 2$ is raised, C^* being defined as C in (1) but replacing Ω by the subset

$$\Omega^* := \{\omega \in \Omega : x^{-1}\omega(x) \text{ is nonincreasing on } (0, 1]\}.$$

The author observes that the challenge of the problem comes from the fact that

$$\sup_{0<x\leqslant 1} \sup_{\omega \in \Omega} \frac{\hat{\omega}(x)}{\omega(x)} = \sup_{0<x\leqslant 1} \sup_{\omega \in \Omega^*} \frac{\hat{\omega}(x)}{\omega(x)} = 2, \tag{2}$$

where $\hat{\omega}$ stands for the least concave majorant of ω.

In the present paper (Section 5), we calculate the value of C^*, which is actually much closer to 1 than to 2 and turns out to be the same that the corresponding constant for the well known Szász–Mirakyan operator over the interval $[0, \infty)$ (see Section 6). Such a coincidence seems to be of a probabilistic nature, since it is closely related with the classical Poisson approximation to the binomial distribution.

The preceding cases also invite to consider what happens with other celebrated operators such as the Baskakov operator, the gamma operator, and the beta operator, for instance. So, we have found it interesting to do a unifying approach by investigating the above mentioned problems in the general setting of operators of probabilistic type (also called Bernstein-type operators). In the next section, under fairly general assumptions, we obtain formulae expressing the best constants in terms of appropriate characteristics of the involved probability measures. Section 3 provides sufficient conditions for the preservation of the classes of moduli of continuity. In Sections 5–9, we apply the general results to the aforementioned five particular examples of operators, and obtain the exact values of the corresponding constants. Some necessary auxiliary results are collected in Section 4.

2. GENERAL FORMULAE

In this section, I will denote either the interval $[0, 1]$ or the interval $[0, \infty)$, Ω and Ω^* will be the classes of moduli of continuity defined as in the preceding section but replacing $[0, 1]$ by I, and L will be a Bernstein-type operator over I, i.e., a positive linear operator allowing for a representation of the form

$$Lf(x) = Ef(Z(x)), \qquad x \in I, \quad f \in \mathscr{L}, \tag{3}$$

where E denotes mathematical expectation, $Z := \{Z(x) \colon x \in I\}$ is an integrable stochastic process taking values in I, and $\mathscr{L}$ stands for the domain of L, that is, the set of all real measurable functions on I for which the right-hand side in (3) makes sense. It should be observed that the integrability of Z guarantees that $\Omega \subset \mathscr{L}$, since we have, for $\omega \in \Omega$ and $x \in I$,

$$L\omega(x) \leqslant 2\omega(EZ(x)). \tag{4}$$

Actually, (4) trivially holds when $EZ(x) = 0$ (since, in our setting, this implies that $P(Z(x) = 0) = 1$, and, therefore, $L\omega(x) = 0$), while, in case that $EZ(x) > 0$, we have by the subadditivity of ω

$$E\omega(Z(x)) \leqslant E\left\lceil \frac{Z(x)}{EZ(x)} \right\rceil \omega(EZ(x)) \leqslant \left(1 + \frac{EZ(x)}{EZ(x)}\right) \omega(EZ(x)),$$

where

$$\lceil a \rceil := \text{the smallest integer not less than } a.$$

In particular, $\mathscr{L}$ contains the monomial $e_1(x) := x$. L is said to be *centered at* $x \in I$, if $Le_1(x) = EZ(x) = x$.

Let $C(x)$ and $C^*(x)$ be defined by

$$C(x) := \sup_{\omega \in \Omega} \frac{L\omega(x)}{\omega(x)}, \qquad 0 < x \in I, \tag{5}$$

and

$$C^*(x) := \sup_{\omega \in \Omega^*} \frac{L\omega(x)}{\omega(x)}, \qquad 0 < x \in I. \tag{6}$$

Our main result in this section is the following theorem giving formulae for these quantities in terms of the probability distribution of $Z(x)$. We denote by 1_A the indicator function of the subset A of the real line.

THEOREM 1. *We have, for* $0 < x \in I$,

$$C(x) = E\left\lceil \frac{Z(x)}{x} \right\rceil, \tag{7}$$

and

$$C^*(x) = P(Z(x) > 0) + \frac{1}{x} E((Z(x) - x)\, 1_{(x, \infty)}(Z(x))). \tag{8}$$

In particular, if L *is centered at* x, *then*

$$C(x) \leqslant 1 + P(Z(x) > 0) \leqslant 2, \tag{9}$$

and

$$C^*(x) = P(Z(x) > 0) + \frac{1}{2x} E\,|Z(x) - x|. \tag{10}$$

Proof. Fix $0 < x \in I$. For each $\omega \in \Omega$, the subadditivity property leads to

$$L\omega(x) = E\omega(Z(x)) \leqslant E\left\lceil \frac{Z(x)}{x} \right\rceil \omega(x),$$

implying that

$$C(x) \leqslant E\left\lceil \frac{Z(x)}{x} \right\rceil.$$

To show the converse inequality, we adapt an argument taken from [3]. For each $0 < \varepsilon < x$, we define $\omega_\varepsilon(\cdot)$ in the following way

$$\omega_\varepsilon(z) := \left\lceil \frac{z}{x} \right\rceil + \sum_{k=0}^{\infty} \left(\frac{z - kx}{\varepsilon} - 1 \right) 1_{(kx,\, kx+\varepsilon)}(z), \qquad z \in I.$$

It is readily checked that $\omega_\varepsilon \in \Omega$. Moreover, we have $\omega_\varepsilon(x) = 1$, and

$$\omega_\varepsilon(\cdot) \uparrow \left\lceil \frac{\cdot}{x} \right\rceil \qquad \text{as} \quad \varepsilon \downarrow 0.$$

We therefore have, by the monotone convergence theorem,

$$C(x) \geqslant \lim_{\varepsilon \downarrow 0} L\omega_\varepsilon(x) = \lim_{\varepsilon \downarrow 0} E\omega_\varepsilon(Z(x)) = E\left\lceil \frac{Z(x)}{x} \right\rceil,$$

completing the proof of (7). Relation (9) immediately follows from (7), the centeredness assumption, and the fact that we have, for every nonnegative random variable U,

$$E\lceil U \rceil \leqslant P(U > 0) + EU.$$

To show (8), let $\omega^* \in \Omega^*$. Using that $\omega^*(0) = 0$, that ω^* is nondecreasing, and the specific property of ω^* as an element of Ω^*, we have

$$\begin{aligned} L\omega^*(x) &= E(\omega^*(Z(x))\, 1_{(0,\,x]}(Z(x))) + E(\omega^*(Z(x))\, 1_{(x,\,\infty)}(Z(x))) \\ &\leqslant \omega^*(x)\, E(1_{(0,\,x]}(Z(x))) + \frac{\omega^*(x)}{x}\, E(Z(x)\, 1_{(x,\,\infty)}(Z(x))) \\ &= \omega^*(x) \Big(E(1_{(0,\,\infty)}(Z(x))) - E(1_{(x,\,\infty)}(Z(x))) \\ &\quad + \frac{1}{x}\, E(Z(x)\, 1_{(x,\,\infty)}(Z(x))) \Big) \\ &= \omega^*(x) \left(P(Z(x) > 0) + \frac{1}{x}\, E((Z(x) - x)\, 1_{(x,\,\infty)}(Z(x))) \right), \end{aligned}$$

implying that

$$C^*(x) \leqslant P(Z(x) > 0) + \frac{1}{x}\, E((Z(x) - x)\, 1_{(x,\,\infty)}(Z(x))).$$

To show the converse inequality, let $0 < \varepsilon < x$, and let $\omega^*_\varepsilon(\cdot)$ be defined by

$$\omega^*_\varepsilon(z) := \frac{z}{\varepsilon}\, 1_{[0,\,\varepsilon)}(z) + 1_{[\varepsilon,\,x]}(z) + \frac{z}{x}\, 1_{(x,\,\infty)}(z), \qquad z \in I. \tag{11}$$

It is readily checked that $\omega^*_\varepsilon \in \Omega^*$. We also have that $\omega^*_\varepsilon(x) = 1$, and

$$\omega^*_\varepsilon(\cdot) \uparrow \omega^*_0(\cdot) \qquad \text{as} \quad \varepsilon \downarrow 0,$$

where

$$\omega^*_0(z) := 1_{(0,\,x]}(z) + \frac{z}{x}\, 1_{(x,\,\infty)}(z), \qquad z \in I.$$

By the monotone convergence theorem, we therefore have

$$C^*(x) \geqslant \lim_{\varepsilon \downarrow 0} L\omega^*_\varepsilon(x) = L\omega^*_0(x) = \text{the right-hand side in (8)},$$

finishing the proof of (8). Finally, (10) follows from (8) and the fact that we have, by the centeredness assumption,

$$E((Z(x) - x)\, 1_{(x,\,\infty)}(Z(x))) = E((x - Z(x))\, 1_{[0,\,x]}(Z(x))) = \tfrac{1}{2}\, E\, |Z(x) - x|.$$

The proof of the theorem is complete. ∎

Remark 1. Against what happens in [11, 4], the notion of a least concave majorant does not play any role in our developments. However, in connection with (2), it is worth noting that, when $I=[0,1]$, the least concave majorant of the modulus ω_ε^* defined in (11) is given by

$$\hat{\omega}_\varepsilon^*(z)=\frac{z}{\varepsilon}\,1_{[0,\varepsilon)}(z)+\frac{x-\varepsilon+(1-x)\,z}{x(1-\varepsilon)}\,1_{[\varepsilon,1]}(z),$$

and we therefore have

$$\sup_{0<\varepsilon<x\leqslant 1}\frac{\hat{\omega}_\varepsilon^*(x)}{\omega_\varepsilon^*(x)}=\sup_{0<\varepsilon<x\leqslant 1}\frac{2x-x^2-\varepsilon}{x(1-\varepsilon)}=2.$$

3. PRESERVATION OF Ω AND Ω^*

In this section I, L, and Z are the same as in (3), and we are interested in conditions guaranteeing the preservation by L of the classes Ω and Ω^*. The following theorem gives sufficient conditions for the preservation of Ω. It is actually (except for the question of continuity) a part of the proof of [3, Theorem 1], and it is included here for the sake of completeness. We use the notation $U\simeq V$ to indicate that the random variables U and V have the same probability distribution.

THEOREM 2. *Assume that the following assumptions are fulfilled:*

(H_1) $Z(0)=0$ *a.s.*

(H_2) $\lim_{x\downarrow 0} EZ(x)=0$.

(H_3) *For all* $0\leqslant x<y\in I$, *we have* $Z(y)-Z(x)\simeq Z(y-x)$.

Then, $L(\Omega)\subset\Omega$.

Proof. Let $\omega\in\Omega$. We have from (H_1) that $L\omega(0)=\omega(0)=0$. On the other hand, let, $0\leqslant x<y\in I$. From (H_3), we have that $Z(x)\leqslant Z(y)$ a.s.; therefore, $L\omega$ is nondecreasing, as ω is. From the subadditivity of ω, and (H_3), we also have

$$\begin{aligned}L\omega(y)&=E\omega(Z(x)+Z(y)-Z(x))\leqslant E\omega(Z(x))+E\omega(Z(y)-Z(x))\\&=L\omega(x)+L\omega(y-x),\end{aligned}$$

showing the subadditivity of $L\omega$. Finally, $L\omega$ is continuous at 0 (by the subadditivity, this already implies that $L\omega$ is continuous at each $x\in I$), since the relation $\lim_{x\downarrow 0} L\omega(x)=0$ follows from (4), (H_2) and the continuity of ω. This completes the proof that $L\omega\in\Omega$. ∎

Remark 2. Condition (H_1) means that L is interpolating at 0. This condition is obviously necessary for the existence of a constant $C>0$ such that

$$L\omega(x) \leqslant C\omega(x), \qquad x \in I, \quad \omega \in \Omega^*.$$

In the presence of (H_1), (H_2) means that Le_1 is continuous at 0, and (H_3) means that the stochastic process Z has stationary increments, but this last condition is much more difficult to characterize in terms of the operator L. Conditions (H_1) and (H_3) correspond to conditions (B) and (C) in [3], and it should be observed that such conditions already imply the assumption (A) in the same paper.

Remark 3. For the operators fulfilling conditions (H_1) and (H_3), it is shown in [3] that

$$C^0(x) := \sup_{f \in \mathcal{M}} \frac{\omega(Lf; x)}{\omega(f; x)} = E\left\lceil \frac{Z(x)}{x} \right\rceil, \qquad 0 < x \in I,$$

where $\omega(f; \cdot)$ stands for the usual modulus of continuity of f, and

$$\mathcal{M} := \{f \in \mathcal{L} : 0 < \omega(f; 1) < \infty\}.$$

In other words, according to (7), we have $C^0(x) = C(x)$ $(0 < x \in I)$. In the same work, the authors calculate the best uniform constants for the concrete operators to be considered in Sections 5–9. For additional results in the same line, and extensions to multivariate operators, we refer to [5–7].

In the following theorem giving conditions for the preservation of the class Ω^*, $E[U \mid V]$ denotes the conditional expectation of the random variable U, given the random variable V.

THEOREM 3. *Assume that Z fulfills conditions* (H_1) *and* (H_2) *above, as well as the following two conditions*

- (H_4) *For all* $0 \leqslant x < y \in I$, *we have* $Z(x) \leqslant Z(y)$ *a.s.*
- (H_5) *For all* $0 \leqslant x < y \in I$, *we have* $E[Z(x) \mid Z(y)] = \frac{x}{y} Z(y)$ *a.s.*

Then, $L(\Omega^*) \subset \Omega^*$.

Proof. Let $\omega \in \Omega^*$. As in the preceding proof, we have that $L\omega(0) = 0$, and that $L\omega$ is nondecreasing (by (H_4)) and continuous at 0. On the other hand, let $0 < x < y \in I$. Using successively the condition $\omega(0) = 0$, (H_4) and the specific property of ω (as a member of Ω^*), the properties of conditional expectation, and (H_5), we obtain

$$
\begin{aligned}
L\omega(x) &= E(\omega(Z(x))\, 1_{(0,\infty)}(Z(x))) \\
&= E\left(\frac{\omega(Z(x))}{Z(x)}\, Z(x)\, 1_{(0,\infty)}(Z(x))\right) \\
&\geqslant E\left(\frac{\omega(Z(y))}{Z(y)}\, Z(x)\, 1_{(0,\infty)}(Z(y))\right) \\
&= E\left(E\left[\frac{\omega(Z(y))}{Z(y)}\, Z(x)\, 1_{(0,\infty)}(Z(y)) \,\middle|\, Z(y)\right]\right) \\
&= E\left(\frac{\omega(Z(y))}{Z(y)}\, 1_{(0,\infty)}(Z(y))\, E[Z(x) \mid Z(y)]\right) \\
&= E\left(\frac{\omega(Z(y))}{Z(y)}\, 1_{(0,\infty)}(Z(y))\, \frac{x}{y}\, Z(y)\right) \\
&= \frac{x}{y}\, L\omega(y),
\end{aligned}
$$

showing that the function $x^{-1}L\omega(x)$ is nonincreasing on $I-\{0\}$. This already entails the subadditivity of $L\omega$, and completes the proof that $L\omega \in \Omega^*$. ▮

Remark 4. In probabilistic terms, condition (H_4) (which is equivalent to saying that L preserves monotonicity) means that the family of random variables $\{Z(x)\colon x \in I\}$ is nondecreasing for the usual *stochastic order*, while (H_5) means that the family $\{x^{-1}Z(x)\colon 0 < x \in I\}$ is nonincreasing for the *convex order* (see [12]).

4. AUXILIARY RESULTS

In Sections 5–9, we will consider several celebrated examples of Bernstein-type operators, all of which are centered at each x in the corresponding interval. The usual analytic definition of each operator will be accompanied by a specific probabilistic representation useful for our purposes. Such representations have been already used in other works; see, for instance, [1–3, 5–7]. In view of what is said in Remark 3 above, we will only discuss the questions concerning Ω^*. Our calculations concerning $C^*(x)$ (recall formula (10)) will be considerably facilitated by the fact that the literature on probability theory provides explicit formulae for the mean deviations of the involved probability measures. In this section, we collect other necessary auxiliary results, some of which may be of interest in its own right.

The symbol $\lfloor\cdot\rfloor$ stands for the integral part (or floor function), and 0^0 is understood as 1.

LEMMA 1. *Let α and α_k $(k=0, 1, 2, \ldots)$ be defined by*

$$\alpha := \sup_{x\geqslant 0} 1+e^{-x}\left(\frac{x^{\lfloor x\rfloor}}{\lfloor x\rfloor!}-1\right), \qquad \alpha_k := \sup_{k\leqslant x<k+1} 1+e^{-x}\left(\frac{x^k}{k!}-1\right).$$

Then,

$$\alpha=\sup_{k\geqslant 0} \alpha_k=\alpha_3=1+e^{-a_3}\frac{a_3^2}{2}=1.18559\ldots, \tag{12}$$

where

$$a_3 := 1+\tfrac{1}{3}(108-27\sqrt{15})^{1/3}+(4+\sqrt{15})^{1/3} \tag{13}$$

is the unique solution to the equation $x^3-3x^2-6=0$ in the interval $[3, 4)$.

Proof. It is clear that $\alpha=\sup_{k\geqslant 0}\alpha_k$, $\alpha_0=1$, and, for $k\geqslant 1$,

$$\alpha_k=1+e^{-a_k}\frac{a_k^{k-1}}{(k-1)!},$$

where a_k is the unique solution to the equation $x^k-kx^{k-1}-k!=0$ in the interval $[k, k+1)$. It is also readily checked that

$$\alpha_1=1+\exp(-2), \qquad \text{and} \qquad \alpha_2=1+\exp(-1-\sqrt{3})(1+\sqrt{3}).$$

By using computational devices, we can obtain the values of a_3 and α_3, as they are given in (13) and (12), respectively, and we also have that

$$a_4=4.30153\ldots, \qquad \text{and} \qquad \alpha_4=1.17971\ldots$$

(the expression by roots of a_4 being too long to be written here). On the other hand, we have, for $k\geqslant 0$,

$$\alpha_k\leqslant\alpha'_k := \sup_{k\leqslant x<k+1} 1+e^{-x}\frac{x^k}{k!}=1+e^{-k}\frac{k^k}{k!}.$$

Since the sequence $\{\alpha'_k: k\geqslant 0\}$ is nonincreasing, and

$$\max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha'_5\}=\alpha_3,$$

the conclusion follows. ∎

LEMMA 2. *For $2 \leqslant k \leqslant x < k+1 \leqslant n$ (k being an integer), we have*

$$\left(1-\frac{x}{n}\right)^{n-1} < e^{-x} < \left(1-\frac{x}{n}\right)^{n-(k+1)} \prod_{i=1}^{k} \left(1-\frac{i}{n}\right). \tag{14}$$

Proof. Let $2 \leqslant k \leqslant x < k+1 \leqslant n$ be fixed. Using the Taylor series of $\log(1+\cdot)$, the logarithm of the right-hand side in (14) is equal to

$$-(n-(k+1)) \sum_{r=1}^{\infty} \frac{x^r}{rn^r} - \sum_{r=1}^{\infty} \frac{1^r+2^r+\cdots+k^r}{rn^r} = -x + \sum_{r=1}^{\infty} \frac{u_r(x)}{r(r+1)\,n^r},$$

where

$$u_r(x) := (r+1)(k+1)\,x^r - rx^{r+1} - (r+1)(1^r+2^r+\cdots+k^r),$$

and the proof of the second inequality in (14) will be complete as soon as we show that

$$u_r(x) > 0, \qquad r \geqslant 1.$$

Since $u_r(\cdot)$ is nondecreasing on $[k, k+1)$ (as it readily follows by differentiation), we actually have, for $r \geqslant 1$,

$$u_r(x) \geqslant u_r(k) = k^{r+1} - (r+1)(1^r+2^r+\cdots+(k-1)^r) > 0,$$

the last inequality because

$$\frac{1^r+2^r+\cdots+(k-1)^r}{k^{r+1}} = \frac{1}{k} \sum_{i=1}^{k-1} \left(\frac{i}{k}\right)^r < \int_0^1 t^r\,dt = \frac{1}{r+1}.$$

Similarly, the first inequality in (14) follows from the fact that

$$(n-1)\log\left(1-\frac{x}{n}\right) = -x - \sum_{r=1}^{\infty} \frac{x^r[rx-(r+1)]}{r(r+1)\,n^r},$$

and $rx-(r+1) \geqslant 0$, for $2 \leqslant x < n$ and $r \geqslant 1$, with equality only when $x=2$ and $r=1$. This completes the proof of Lemma 2. ∎

LEMMA 3. *We have, for $2 \leqslant x < n$,*

$$\binom{n-1}{\lfloor x \rfloor} \left(\frac{x}{n}\right)^{\lfloor x \rfloor} \left(1-\frac{x}{n}\right)^{n-\lfloor x \rfloor} - \left(1-\frac{x}{n}\right)^n < e^{-x} \left(\frac{x^{\lfloor x \rfloor}}{\lfloor x \rfloor !} - 1\right). \tag{15}$$

Proof. Denote by $f_n(x)$ the difference between the right-hand side and the left-hand side in (15). We have from Lemma 2

$$f_n(2)=e^{-2}-\left(1-\frac{2}{n}\right)^{n-1}>0.$$

To achieve the result, it therefore suffices to show that $f_n(\cdot)$ is increasing on $[2, n)$. Since such a function is continuous, this reduces to show that it has a positive derivative on the interval $(k, k+1)$, for each integer $k\in[2, n)$. Fix the integer k. For $x\in(k, k+1)$, we can write

$$f_n(x)=g_n(x)+h_n(x),$$

where

$$g_n(x):=\left(1-\frac{x}{n}\right)^n-e^{-x},$$

and

$$h_n(x):=\left[e^{-x}-\left(1-\frac{x}{n}\right)^{n-k}\prod_{i=1}^{k}\left(1-\frac{i}{n}\right)\right]\frac{x^k}{k!}.$$

Since

$$g_n'(x)=e^{-x}-\left(1-\frac{x}{n}\right)^{n-1},$$

and

$$h_n'(x)=\left[\left(1-\frac{x}{n}\right)^{n-(k+1)}\prod_{i=1}^{k}\left(1-\frac{i}{n}\right)-e^{-x}\right]\frac{x^{k-1}}{(k-1)!}\left(\frac{x}{k}-1\right),$$

the conclusion follows from Lemma 2. ∎

Remark 5. From the proof of Lemma 2, it clearly follows that the left-hand side (resp., the right-hand side) in (14) is increasing (resp., decreasing) as n increases (k and x remaining fixed). We therefore have

$$K_n(x)<K_{n+1}(x), \qquad 2\leqslant x<n,$$

where $K_n(x)$ is the left-hand side in (15), as it follows by the same argument showing Lemma 3.

5. BERNSTEIN OPERATORS

For each $n = 1, 2, \ldots$, the Bernstein operator B_n over the interval $[0, 1]$ given in the introduction allows for the representation

$$B_n f(x) = Ef\left(\frac{S_n(x)}{n}\right),$$

where

$$S_n(x) := \sum_{i=1}^{n} 1_{[0,x]}(X_i),$$

and $X_1, X_2, \ldots$ are independent and on the interval $[0, 1]$ identically distributed random variables, so that the random variable $S_n(x)$ has the binomial distribution with parameters n and x.

For these operators, we have the following results, where we write $C_n^*(x)$ instead of $C^*(x)$ for the quantity defined in (6) corresponding to the operator B_n (the same convention will be used in Sections 6–9 below).

THEOREM 4. *The following assertions hold:*

(a) $B_n(\Omega^*) \subset \Omega^*$ $(n \geqslant 1)$.

(b) $C_n^*(x) = 1-(1-x)^n + \binom{n-1}{\lfloor nx \rfloor} x^{\lfloor nx \rfloor}(1-x)^{n-\lfloor nx \rfloor}$ $(0 < x \leqslant 1,\ n \geqslant 1)$.

(c) $\sup_{0<x\leqslant 1} \sup_{n \geqslant 1} C_n^*(x) = \alpha$, *where* α *is the same as in Lemma* 1.

Proof. Part (a) follows from Theorem 3. In fact, conditions (H_1), (H_2), and (H_4) are immediately verified, and, for (H_5), we can refer to [1, pp. 133–134]. An alternative analytic proof runs as follows: For $\omega \in \Omega^*$, $x \in (0, 1]$, and $n \geqslant 2$ (the case $n = 1$ is trivial), we have (using the same notations as in the Introduction)

$$\frac{d}{dx}\left(\frac{B_n\omega(x)}{x}\right) = (n-1)\sum_{k=0}^{n-2}\left[\omega\left(\frac{k+2}{n}\right)\frac{n}{k+2} - \omega\left(\frac{k+1}{n}\right)\frac{n}{k+1}\right] p_{n-2,k}(x)$$
$$\leqslant 0.$$

Part (b) directly follows from (10), by using the fact that

$$P(S_n(x) > 0) = 1-(1-x)^n,$$

and the well known formula for the mean deviation of the binomial distribution (cf. [10, (3.15)])

$$E\,|S_n(x) - nx| = 2nx\binom{n-1}{\lfloor nx \rfloor} x^{\lfloor nx \rfloor}(1-x)^{n-\lfloor nx \rfloor}.$$

From part (b), we have that

$$\lim_{n\to\infty} C_n^*(x/n) = 1 + e^{-x}\left(\frac{x^{\lfloor x\rfloor}}{\lfloor x\rfloor!} - 1\right), \qquad x > 0,$$

showing that

$$\sup_{0<x\leqslant 1}\ \sup_{n\geqslant 1} C_n^*(x) \geqslant \alpha,$$

and the proof of the theorem will be complete as soon as we show that

$$\sup_{0<x\leqslant 1}\ \sup_{n\geqslant 1} C_n^*(x) \leqslant \alpha. \tag{16}$$

First, we obviously have

$$C_n^*(x) = 1 = C_n^*(1), \qquad n\geqslant 1, \quad x\in(0, 1/n). \tag{17}$$

Second, it is readily checked that, for $n\geqslant 2$, the maximum of the function $C_n^*(\cdot)$ on the interval $[1/n, 2/n)$ is achieved at the point $x_n := 2n^{-1} - n^{-2}$. We therefore have, by Lemma 2 and Lemma 1,

$$\sup_{1/n\leqslant x<2/n} C_n^*(x) = C_n^*(x_n) = 1 + \left(1 - \frac{2}{2n}\right)^{2n-1} < 1 + e^{-2} < \alpha. \tag{18}$$

Finally, we directly have from Lemma 3 and Lemma 1

$$C_n^*(x) < 1 + e^{-nx}\left(\frac{(nx)^{\lfloor nx\rfloor}}{\lfloor nx\rfloor!} - 1\right) \leqslant \alpha, \qquad n\geqslant 3, \quad 2/n\leqslant x<1, \tag{19}$$

and the inequality (16) follows from (17)–(19). ∎

6. SZÁSZ–MIRAKYAN OPERATORS

For $t>0$, the Szász–Mirakyan operator S_t over $[0, \infty)$ is defined by

$$S_t f(x) := \sum_{k=0}^{\infty} f(k/t)\,\pi_{t,k}(x) = Ef\left(\frac{N(tx)}{t}\right), \tag{20}$$

where the $\pi_{t,k}(x)$ are the weights of the Poisson distribution with parameter tx

$$\pi_{t,k}(x) := e^{-tx}\frac{(tx)^k}{k!},$$

and $\{N(u)\colon u\geqslant 0\}$ is a standard Poisson process.

For these operators, we assert the following.

THEOREM 5. *We have:*

(a) $S_t(\Omega^*) \subset \Omega^*$ $(t > 0)$.

(b) $C_t^*(x) = 1 + e^{-tx}(\frac{(tx)^{\lfloor tx \rfloor}}{\lfloor tx \rfloor !} - 1)$ $(x > 0, t > 0)$.

(c) $\sup_{x>0} C_t^*(x) = \alpha$ $(t > 0)$, *where* α *is the same as in Lemma* 1.

Proof. Part (a) can be shown by using the same probabilistic or analytic arguments as in the proof of Theorem 4(a). Part (b) follows from (20), (10), and the fact that we have for t, $x > 0$,

$$P(N(tx) > 0) = 1 - \pi_{t,0}(x) = 1 - e^{-tx},$$

as well as (cf. [10, (4.19)])

$$E\,|N(tx) - tx| = 2txe^{-tx} \frac{(tx)^{\lfloor tx \rfloor}}{\lfloor tx \rfloor !}.$$

Finally, part (c) is an immediate consequence of part (b) and Lemma 1. ∎

Remark 6. It should be remarked that the best uniform constant $\sup_{x>0} C_t^*(x)$ does not depend upon t (as it could be expected in advance from the very definition of S_t), as well as the coincidence (already mentioned in the Introduction) with the best uniform constant for the family of the Bernstein operators.

7. GAMMA OPERATORS

For $t > 0$, the gamma operator G_t over the interval $[0, \infty)$ is the integral operator defined by

$$G_t f(x) := \frac{1}{\Gamma(t)} \int_0^\infty f\left(\frac{x\theta}{t}\right) \theta^{t-1} e^{-\theta}\, d\theta = Ef\left(\frac{xU_t}{t}\right),$$

where $\{U_t : t \geqslant 0\}$ is a standard gamma process, i.e., a stochastic process starting at 0, having stationary independent increments, and such that, for each $t > 0$, the random variable U_t has the gamma distribution with density

$$g_t(\theta) := \frac{\theta^{t-1} e^{-\theta}}{\Gamma(t)} 1_{(0,\infty)}(\theta). \tag{21}$$

For these operators, we can assert the following.

THEOREM 6. *We have:*

(a) $G_t(\Omega^*) \subset \Omega^*$ $(t>0)$.

(b) $C_t^*(x) = 1 + g_t(t)$ $(t, x > 0)$, *where* g_t *is given in* (21).

(c) $\sup_{t,x>0} C_t^*(x) = 2$.

Proof. Part (a) is shown in a very easy way. Actually, for $\omega \in \Omega^*$, it is immediate that $G_t\omega(\cdot)$ is a nondecreasing continuous function on the semi-axis vanishing at 0, and we have, for $0 < x < y$,

$$G_t\omega(x) = E\omega\left(\frac{xU_t}{t}\right) \geqslant \frac{x}{y}\, E\omega\left(\frac{yU_t}{t}\right) = \frac{x}{y}\, G_t\omega(y).$$

From (10), we have for t, $x > 0$

$$C_t^*(x) = P(U_t > 0) + \frac{1}{2}\, E\left|\frac{U_t}{t} - 1\right| = 1 + \frac{1}{2}\, E\left|\frac{U_t}{t} - 1\right|, \tag{22}$$

and the relation (cf. [8, (17.12)])

$$E\,|U_t - t| = \frac{2t^t e^{-t}}{\Gamma(t)}$$

yields part (b). Finally, we have

$$2 = 1 + \lim_{t \downarrow 0} g_t(t) \leqslant \sup_{t,x>0} C_t^*(x) \leqslant 2$$

(the last inequality by (9)), showing part (c) and completing the proof of the theorem. ∎

Remark 7. It is worth noting that Theorem 6(c) actually provides a new proof for the fact (first proved in [3]) that $\sup_{t,x>0} C_t^0(x) = 2$, where $C_t^0(x)$ is the quantity (corresponding to the operator G_t) defined in Remark 3 above.

8. BASKAKOV OPERATORS

For $t > 0$, the Baskakov operator H_t over $[0, \infty)$ is defined by

$$H_t f(x) := \sum_{k=0}^{\infty} f(k/t)\, b_{t,k}(x) = E\left(\frac{N(xU_t)}{t}\right),$$

where the $b_{t,k}(x)$ are the weights of the negative binomial distribution with parameters t, x, i.e.,

$$b_{t,k}(x) := \binom{t+k-1}{k} \frac{x^k}{(1+x)^{t+k}},$$

$\{N(u) \colon u \geqslant 0\}$ is a standard Poisson process, and $\{U_t \colon t \geqslant 0\}$ is a standard gamma process independent of the former.

For these operators, we establish the following.

THEOREM 7. *We have:*

(a) $H_t(\Omega^*) \subset \Omega^*$ $(t > 0)$.

(b) $C_t^*(x) = 1 - (1+x)^{-t} + \binom{t+\lfloor tx \rfloor}{\lfloor tx \rfloor} \frac{x^{\lfloor tx \rfloor}}{(1+x)^{t+\lfloor tx \rfloor}}$ $(t, x > 0)$.

(c) $\sup_{t, x > 0} C_t^*(x) = 2$.

Proof. Part (a) is shown in the same probabilistic or analytic way that Theorem 4(a) or Theorem 5(a), and we omit the details. Part (b) follows from (10) and the facts that, for $t, x > 0$,

$$P(N(xU_t) > 0) = 1 - b_{t,0}(x) = 1 - (1+x)^{-t},$$

and (cf. [10, (5.26)])

$$E\,|N(xU_t) - tx| = 2tx \binom{t+\lfloor tx \rfloor}{\lfloor tx \rfloor} \frac{x^{\lfloor tx \rfloor}}{(1+x)^{t+\lfloor tx \rfloor}}.$$

Finally, we show part (c). First, we observe that, by the strong law of large numbers for the standard Poisson process, we have

$$\lim_{x \to \infty} \frac{N(xU_t)}{xt} = \frac{U_t}{t} \text{ a.s.,} \qquad t > 0.$$

Using this fact, (10), and Fatou's lemma, we therefore have, for $t > 0$,

$$\sup_{x > 0} C_t^*(x) \geqslant \liminf_{x \to \infty} C_t^*(x) \geqslant 1 + \frac{1}{2} E \left| \frac{U_t}{t} - 1 \right| = 1 + g_t(t),$$

where $g_t(\cdot)$ is given in (21). Thus, part (c) follows from (9) and Theorem 6(b,c). ∎

Remark 8. Remark 7 above remains true if G_t and Theorem 6(c) are replaced by H_t and Theorem 7(c), respectively.

9. BETA OPERATORS

For $t>0$, the beta operator β_t over the interval $[0, 1]$ is defined by

$$\beta_t f(x) := \begin{cases} \int_0^1 f(\theta)\, b_t(x;\theta)\, d\theta) & \text{if} \quad x \in (0, 1) \\ f(x) & \text{if} \quad x = 0 \quad \text{or} \quad 1 \end{cases}$$

$$= Ef\left(\frac{U_{tx}}{U_t}\right),$$

where $b_t(x;\cdot)$ is the density of the beta distribution with parameters tx, $t(1-x)$, i.e.,

$$b_t(x;\theta) := \frac{\theta^{tx-1}(1-\theta)^{t(1-x)-1}}{B(tx, t(1-x))}\, 1_{(0,1)}(\theta) \tag{22}$$

($B(\cdot,\cdot)$ being the beta function), and $\{U_t : t>0\}$ is the same standard gamma process as is Section 7. It should be observed that, since the operator β_t interpolates at 1, we have $C_t^*(1)=1$.

The main results for these operators are collected in the following theorem.

THEOREM 8. *We have:*

(a) $\beta_t(\Omega^*) \subset \Omega^*$ $(t>0)$.

(b) $C_t^*(x) = 1 + \frac{1-x}{t}\, b_t(x;x)$ $(t>0,\ x\in(0,1))$, *where* $b_t(\cdot\,;\cdot)$ *is given in* (22).

(c) $\sup_{0<x<1} C_t^*(x) = 2$ $(t>0)$.

Proof. Part (a) follows by applying Theorem 3 (conditions (H_1), (H_2), and (H_4) are immediately verified, and, for (H_5), we refer to [2, p. 4]). Part (b) follows from (10), the fact that

$$P\left(\frac{U_{tx}}{U_t} > 0\right) = 1,$$

and the following formula for the mean deviation of the beta distribution (cf. [9, (25.18a)])

$$E\left|\frac{U_{tx}}{U_t} - x\right| = \frac{2}{t}\, \frac{x^{tx}(1-x)^{t(1-x)}}{B(tx, t(1-x))}.$$

Finally, using part (b), it is readily checked that, for $t > 0$,

$$\sup_{0<x<1} C_t^*(x) \geqslant \lim_{x\downarrow 0} C_t^*(x) = 2,$$

which, together with (9), yields part (c). This completes the proof of the theorem. ∎

Remark 9. The observations in Remarks 7 and 8 also hold for beta operators.

REFERENCES

1. J. A. Adell and J. de la Cal, Using stochastic processes for studying Bernstein-type operators, *Rend. Circ. Mat. Palermo (2) Suppl.* **33** (1993), 125–141.
2. J. A. Adell and J. de la Cal, Beta-type operators preserve shape properties, *Stochastic Process. Appl.* **48** (1993), 1–8.
3. J. A. Adell and A. Pérez-Palomares, Best constants in preservation inequalities concerning the first modulus and Lipschitz classes for Bernstein-type operators, *J. Approx. Theory* **93** (1998), 128–139.
4. G. A. Anastassiou, C. Cottin, and H. H. Gonska, Global smoothness of approximating functions, *Analysis* **11** (1991), 43–57.
5. J. de la Cal and A. M. Valle, Best constants in global smoothness preservation inequalities for some multivariate operators, *J. Approx. Theory* **97** (1999), 158–180.
6. J. de la Cal and A. M. Valle, Global smoothness preservation by multivariate Bernstein-type operators, *in* "Handbook on Analytic Computational Methods in Applied Mathematics" (G. A. Anastassiou, Ed.), pp. 667–707, CRC Press, Boca Raton, FL, 2000.
7. J. de la Cal and A. M. Valle, Best constants for tensor products of Bernstein, Szász and Baskakov operators, *Bull. Austral. Math. Soc.* **62** (2000), 211–220.
8. N. L. Johnson, S. Kotz, and N. Balakrishnan, "Continuous Univariate Distributions," Vol. 1, 2nd ed., Wiley, New York, 1994.
9. N. L. Johnson, S. Kotz, and N. Balakrishnan, "Continuous Univariate Distributions," Vol. 2, 2nd ed., Wiley, New York, 1995.
10. N. L. Johnson, S. Kotz, and A. W. Kemp, "Univariate Discrete Distributions," 2nd ed., Wiley, New York, 1992.
11. Z. Li, Bernstein polynomials and modulus of continuity, *J. Approx. Theory* **102** (2000), 171–174.
12. M. Shaked and J. G. Shanthikumar, "Stochastic Orders and Their Applications," Academic Press, Boston, 1994.